On Finite Difference Methods of Solution of the Transport Equation

By R. P. Pearce and A. R. Mitchell

1. Introduction. In recent years several difference schemes have been proposed for solving the transport equation

$$\frac{\partial u}{\partial t} + V(x, t, u)\frac{\partial u}{\partial x} = F(x, t, u) \tag{1}$$

in one form or another, where V is the velocity of propagation of a profile given initially along the x-axis. Most of these schemes can be found in Richtmyer [1] and, generally speaking, they are chosen primarily from the point of view of stability.

An equation of the type (1) has a single family of characteristics in the (x, t) plane and in any step-by-step method of solution it is essential from the point of view of accuracy that the characteristics be followed as closely as possible. It is proposed to examine existing difference schemes from this standpoint and to derive new formulas of greater accuracy. For the purposes of this paper, it is sufficient to consider the simplified version of (1)

$$\frac{\partial u}{\partial t} + V\frac{\partial u}{\partial x} = 0, \tag{2}$$

where V is constant, from which it follows that the given profile at $t = 0$ is propagated without change of shape in the direction of the x-axis with velocity V. If a difference scheme fails to give an accurate solution of (2), it is pointless to consider it as a means of solving more complicated forms of (1), in particular, forms which incorporate variable velocity of propagation and source or sink terms. On the other hand, it is realized that schemes which successfully solve (2) may not give comparable accuracy when used to solve (1). In the case of (1), the characteristics are curved and can only be determined by integration of the equation $\frac{dx}{dt} = V(x, t, u)$. In addition, the equation $\frac{du}{dt} = F(x, t, u)$ has to be solved. These computations, however, involve only numerical integration, a process which can be made as accurate as required in most problems.

2. Stable Finite Difference Schemes Now in Use. Existing stable difference schemes will now be discussed with reference to equation (2). The characteristics of the latter are straight lines inclined to the t-axis at an angle

$$\theta = \tan^{-1} V. \tag{3}$$

In these schemes, the parameter p is introduced where $p = \frac{V\Delta t}{\Delta x}$, and Δx and Δt are the respective mesh lengths in the x and t directions.

Received January 10, 1961.

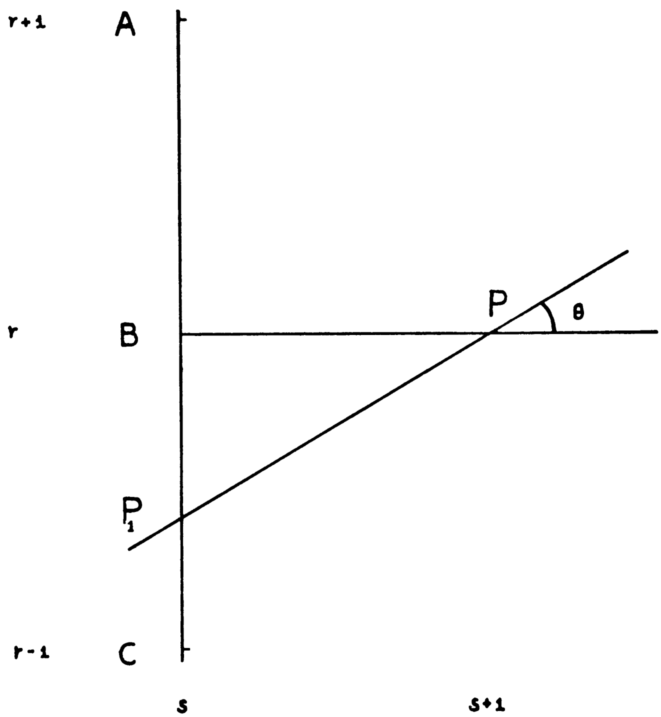

FIG. 1

Difference System I (*Friedrichs* [2]). This is given by

$$u_{r,s+1} = \tfrac{1}{2}(1 - p)u_{r+1,s} + \tfrac{1}{2}(1 + p)u_{r-1,s} \tag{4}$$

where $x = r\Delta x$ and $t = s\Delta t$. This system can be obtained by replacing $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial t}$ at the node (r, s) by $\frac{1}{2\Delta x}(u_{r+1,s} - u_{r-1,s})$ and $\frac{1}{\Delta t}(u_{r,s+1} - u_{r,s})$ respectively, then substituting $\frac{1}{2}(u_{r+1,s} + u_{r-1,s})$ for $u_{r,s}$. Another and more satisfactory way of deriving (4) is now proposed. In Figure 1, the characteristic through P cuts AC in P_1 where $BP_1 = p\Delta x$, and it follows that $u_P = u_{P_1}$. Since P_1 is not a mesh point, the value of u at P_1 may be obtained by linear interpolation between A and C, and so (4) is obtained. In addition, since the coefficients on the right-hand side of (4) have sum unity, the solution computed by (4) is bounded if both coefficients are positive which leads immediately to the condition $|p| \leqq 1$ for stability (Richtmyer [1], p. 43).

Difference System II (*Carlson* [3]). This system is given by

$$u_{r,s+1} = (1 - p)u_{r,s} + pu_{r-1,s} \qquad (0 \leqq p \leqq 1) \tag{5a}$$

$$u_{r,s+1} = \frac{1}{1 + p}\,u_{r,s} + \frac{p}{1 + p}\,u_{r-1,s+1} \qquad (p > 1) \tag{5b}$$

and two similar formulas if $p < 0$.

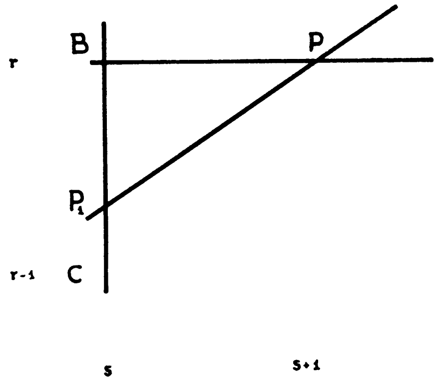

FIG. 2(a)

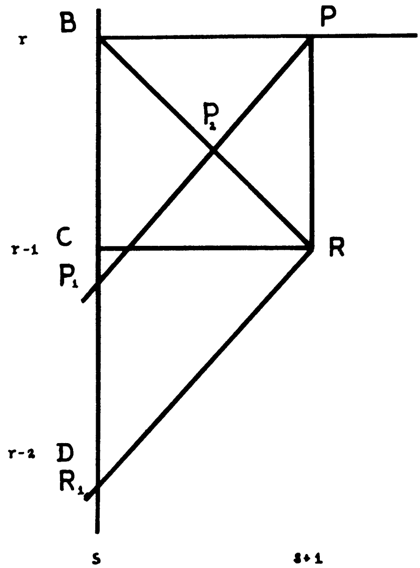

FIG. 2(b)

In Figure 2, PP_1 is the characteristic through P. In this scheme, three points only are used, the choice of points depending on the position of P_1. If $0 \leqq p \leqq 1$, P_1 lies between B and C and the formula used is (5a), whereas if $p > 1$, P_1 lies outside BC and the formula used is (5b). It is presumed that these formulas were obtained originally by replacing $\frac{\partial u}{\partial t}$ by $\frac{1}{\Delta t}(u_{r,s+1} - u_{r,s})$ and $\frac{\partial u}{\partial x}$ by $\frac{1}{\Delta x}(u_{r,s} - u_{r-1,s})$ or $\frac{1}{\Delta x}(u_{r,s+1} - u_{r-1,s+1})$ for $0 \leqq p \leqq 1$ and $p > 1$ respectively.

When P_1 lies between B and C (Figure 2a) it follows that $BP_1:P_1C = p:1 - p$. Thus, on using linear interpolation between B and C together with the result $u_P = u_{P_1}$, formula (5a) is obtained. When P_1 lies beyond C (Figure 2b) it can be shown that $BP_2:P_2R = p:1$, and so using linear interpolation between B and R together with $u_P = u_{P_2}$, formula (5b) is obtained. The solution computed by (5) is bounded for all p, as the right-hand sides of both (5a) and (5b) sum to unity and have positive coefficients.

As Carlson's scheme has been used extensively to solve problems involving the transport equation [1], [4], it is worth studying in some detail with a view to determining its probable accuracy. If $0 \leqq p \leqq 1$, formula (5a) is as accurate as the linear interpolation of u between B and C. As these are neighboring mesh points on $t = s\Delta t$, the line of most recently computed values of u, it is to be expected that (5a) will give reasonably accurate values of u. Certainly (5a) will be superior to scheme (4) proposed by Friedrichs since the latter uses linear interpolation of u between A and C, mesh points two distance intervals apart. If $p > 1$, however, a much less satisfactory state of affairs exists. In Figure 2b, RR_1 is the characteristic through R, and theoretically $u_R = u_{R_1}$. Similarly, $u_P = u_{P_1}$ and (5b) becomes

$$u_{P_1} = \frac{1}{1+p} u_B + \frac{p}{1+p} u_{R_1}.$$

Since $BP_1:P_1R_1 = p:1$, this formula is equivalent to linear interpolation between B and R_1 which for large values of p, where B and R_1 are many distance intervals apart, may be very inaccurate. In fact, the foregoing seems to suggest that implicit schemes in general are poor, particularly for large values of p.

Difference System III (Central Difference Formula). This is given by

$$(6) \qquad u_{r,s+1} = u_{r,s-1} - pu_{r+1,s} + pu_{r-1,s},$$

and has been used with success by Malkus and Witt [5] to solve some problems in meteorology involving transport of temperature and vorticity in two dimensions. It is obtained by replacing $\frac{\partial u}{\partial t}$ by $\frac{1}{2\Delta t}(u_{r,s+1} - u_{r,s-1})$ and $\frac{\partial u}{\partial x}$ by $\frac{1}{2\Delta x}(u_{r+1,s} - u_{r-1,s})$. Alternatively, from Figure 3, if GG_1 and PP_1 are the characteristics through G and P respectively, so that $u_G = u_{G_1}$ and $u_P = u_{P_1}$, (6) may be written as

$$u_{P_1} = u_{G_1} - pu_A + pu_C.$$

This result is equivalent to using a parabolic interpolation formula incorporating values of u at A, G_1, and C, thus (6) is expected to be an accurate formula, particularly for small values of $|p|$. In fact, (6) is stable for $-1 \leqq p \leqq 1$, and can only be used if G_1 and P_1 lie between A and C.

It is interesting to compare the foregoing predictions of accuracy with numerical calculations carried out using difference systems I, II, and III in turn to solve (2). Two initial profiles of u are considered, the "roof top" and the "sine" and these are illustrated in Figure 4. All calculations are carried out until a time $\frac{6\Delta x}{V}$ is reached.

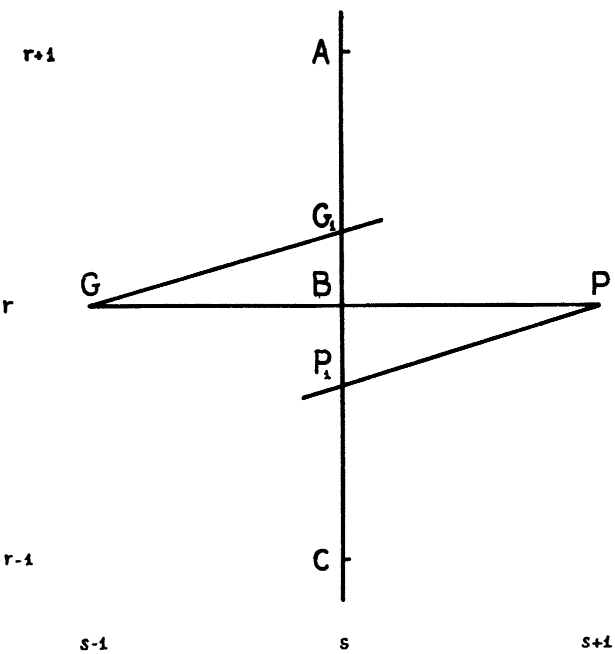

FIG. 3

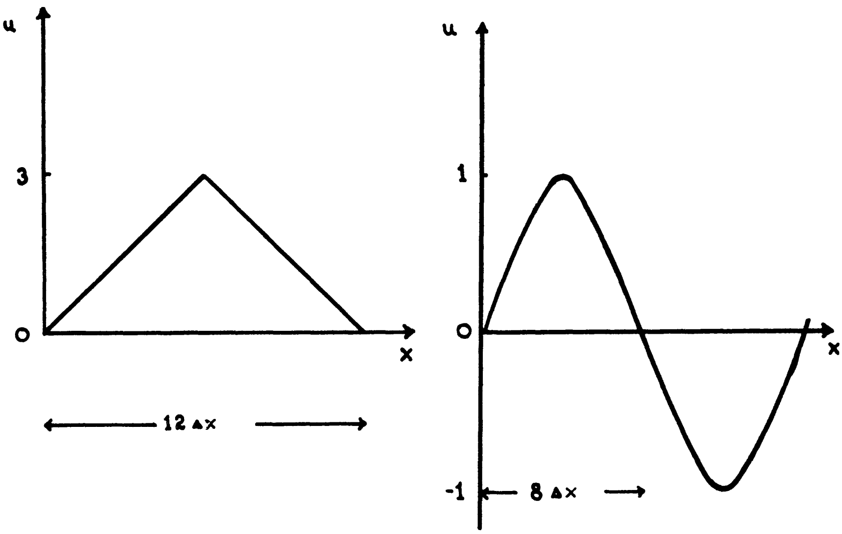

FIG. 4

Theoretical values are used at the second time step in order to start the calculation using System III. The results, accurate to 0.001, are shown in Tables 1(a) and 1(b) for the "roof top" and "sine" profiles respectively. The last row of these tables gives the sum of the moduli of the errors $\sum | e |$. The outstanding features of these results are the poor accuracy of Carlson's scheme for $| p | > 1$, and the comparatively high accuracy of the central difference formula.

3. Two-Level Interpolation Schemes. As a consequence of the last section, explicit difference schemes which are high accuracy interpolation formulas seem most likely to succeed in obtaining accurate solutions of the transport equation. With this in mind, several new two-level formulas are now proposed and used to solve (2). These formulas give $u_{r,s+1}$ in terms of u at nodes on the time step s.

I. Linear Interpolation Formulas.

(7a) $$u_{r,s+1} = (1 - p)u_{r,s} + pu_{r-1,s} \qquad (0 \leqq p \leqq 1)$$

(7b) $$u_{r,s+1} = (2 - p)u_{r-1,s} + (p - 1)u_{r-2,s} \qquad (1 \leqq p \leqq 2)$$

(7c) $$u_{r,s+1} = (n + 1 - p)u_{r-n,s} + (p - n)u_{r-n-1,s} \qquad (n \leqq p \leqq n + 1).$$

These formulas are obtained in the following manner using Figure 5. If PP_1 is the characteristic through P so that $u_P = u_{P_1}$ and P_1 lies between B and C, then $BP_1:P_1C = p:1 - p$, and by using linear interpolation of u between B and C, formula (7a) is obtained. If the characteristic through P cuts the line s in P_2 between C and D, then $CP_2:P_2D = p - 1:2 - p$, and linear interpolation of u between C and D gives formula (7b). A similar method may be used for values of p greater than 2.

The general result for any value of p lying between integers n and $n + 1$, where n may be positive or negative, is given by (7c). The formulas are stable since the coefficients on the right-hand sides are positive and add to unity in corresponding pairs.

TABLE 1(a)

r	Correct Values	Scheme: Friedrichs	Carlson		Central Difference		
p		½	½	3	¼	½	¾
s		12	12	2	24	12	8
−10	0	—	—	—	0.001	0	0
−9	0	—	—	—	−0.005	−0.003	0
−8	0	—	—	—	0.007	0.002	0
−7	0	—	—	—	−0.028	−0.024	−0.016
−6	0	—	—	—	0.029	0.015	0.005
−5	0	—	—	—	−0.051	−0.045	−0.033
−4	0	0.001	0	—	0.012	0.006	0.001
−3	0	0.003	0	—	0.053	0.052	0.047
−2	0	0.010	0	—	−0.059	−0.032	−0.010
−1	0	0.017	0	—	0.027	0.006	−0.024
0	0	0.044	0	0.032	0.008	0.014	0.008
1	0	0.071	0.001	0.110	0.166	0.166	0.153
2	0	0.147	0.011	0.241	−0.105	−0.067	−0.024
3	0	0.223	0.047	0.424	−0.270	−0.267	−0.239
4	0	0.384	0.144	0.657	0.021	−0.022	−0.022
5	0	0.545	0.338	0.934	0.200	0.213	0.215
6	0.5	0.796	0.644	1.188	0.465	0.490	0.515
7	1.0	1.046	1.044	1.382	0.842	0.816	0.793
8	1.5	1.311	1.488	1.498	1.676	1.620	1.546
9	2.0	1.575	1.906	1.532	2.403	2.388	2.334
10	2.5	1.715	2.210	1.487	2.605	2.592	2.577
11	3.0	1.856	2.323	1.368	2.631	2.652	2.708
12	2.5	1.773	2.210	1.216	2.426	2.413	2.419
13	2.0	1.691	1.906	1.055	2.023	2.050	2.086
14	1.5	1.428	1.488	0.898	1.408	1.436	1.477
15	1.0	1.166	1.044	0.753	0.817	0.828	0.856
16	0.5	0.873	0.644	0.625	0.424	0.440	0.456
17	0	0.580	0.338	0.514	0.169	0.152	0.116
18	0	0.385	0.144	0.419	0.070	0.066	0.050
19	0	0.190	0.047	0.339	0.019	0.012	0
20	0	0.110	0.011	0.274	0.006	0.004	0
21	0	0.031	0.001	0.219	0	0	0
22	0	0.015	0	0.175	—	—	—
23	0	0	0	0.139	—	—	—
24	0	0	0	0.110	—	—	—
25	0	0	0	0.087	—	—	—
26	0	0	0	0.069	—	—	—
27	0	0	0	0.054	—	—	—
28	0	—	—	0.042	—	—	—
29	0	—	—	0.033	—	—	—
30	0	—	—	0.026	—	—	—
31	0	—	—	0.020	—	—	—
$\sum \lvert e \rvert$		7.298	2.907	12.308	3.000	2.723	2.312

TABLE 1(b)

r	Correct Values	Scheme: Friedrichs	Carlson		Central Difference
	p	$\frac{1}{2}$	$\frac{1}{2}$	3	$\frac{1}{2}$
	s	12	12	2	12
0	−0.707	−0.317	−0.560	−0.477	−0.671
1	−0.924	−0.439	−0.731	−0.388	−0.897
2	−1.000	−0.495	−0.792	−0.269	−0.996
3	−0.924	−0.475	−0.731	−0.128	−0.938
4	−0.707	−0.383	−0.560	0.022	−0.738
5	−0.383	−0.232	−0.303	0.162	−0.430
6	0	−0.046	0	0.276	−0.047
7	0.383	0.146	0.303	0.346	0.350
8	0.707	0.317	0.560	0.364	0.671
9	0.924	0.439	0.732	0.327	0.897
10	1.000	0.495	0.792	0.242	0.996
11	0.924	0.475	0.732	0.121	0.938
12	0.707	0.383	0.560	−0.016	0.738
13	0.383	0.232	0.303	−0.150	0.430
14	0	0.046	0	−0.261	0.047
15	−0.383	−0.146	−0.303	−0.330	−0.350
$\sum \mid e \mid$		5.174	2.094	7.950	0.478

II. Parabolic Interpolation Formulas.

$$(8)\quad u_{r,s+1} = -\tfrac{1}{2}(p-n)(n+1-p)u_{r+1-n,s} + (p-n+1)(n+1-p)u_{r-n,s} + \tfrac{1}{2}(p-n)(p-n+1)u_{r-1-n,s} \qquad (n \leqq p \leqq n+1).$$

Referring again to Figure 5, if PP_1 is the characteristic through P, where P_1 lies between B and C, and if a parabolic interpolation formula incorporating the values of u at A, B and C is used to give u at points between B and C then $u_{r,s+1}$ is given by (8) with $n = 0$. If the characteristic through P cuts the line s at P_2 where P_2 lies between C and D, and a parabolic interpolation formula incorporating the values of u at B, C, and D is used to give u at points between C and D, then $u_{r,s+1}$ is given by (8) with $n = 1$, and so on for higher values of n. Finally, the stability of (8) is easily demonstrated by using methods described in Richtmyer [1], since the equations are linear and have constant coefficients. Other stable parabolic interpolation schemes based on (8) are possible but they are unlikely to be more accurate than (8) with the original range of p stated.

III. Cubic Interpolation Formulas.

$$(9)\quad \begin{aligned} u_{r,s+1} = & -\tfrac{1}{6}(p-n)(n+1-p)(n+2-p)u_{r+1-n,s} \\ & + \tfrac{1}{2}(n+2-p)(n+1-p)(p+1-n)u_{r-n,s} \\ & + \tfrac{1}{2}(p-n)(n+2-p)(p+1-n)u_{r-1-n,s} \\ & - \tfrac{1}{6}(p-n)(n+1-p)(p+1-n)u_{r-2-n,s} \quad (n \leqq p \leqq n+1). \end{aligned}$$

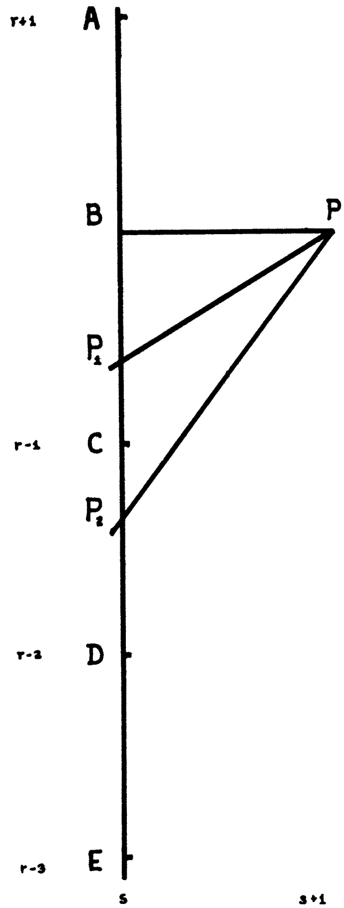

FIG. 5

From Figure 5, if P_1 lies between B and C, and a cubic interpolation formula based on the values of u at A, B, C, and D is used to give values of u between B and C, then $u_{r,s+1}$ is given by (9) with $n = 0$. If, however, PP_2 is the characteristic through P where P_2 lies between C and D, and a cubic interpolation formula based on the values of u at B, C, D and E is used to give values of u between C and D, then $u_{r,s+1}$ is given by (9) with $n = 1$, and so on. Formula (9) is stable not only for the range of p stated but for the extended range $n - 1 \leqq p \leqq n + 2$, and so other cubic interpolation schemes based on (9) are possible. One other possible scheme is (9) together with $n - 1 \leqq p \leqq n + 2$ for $n = \cdots -6, -3, 0, 3, 6, \cdots$. It is unlikely, however, that any of the other schemes will be as accurate as (9) with the original range of p stated and n any integer.

4. Three-Level Formulas. So far the only three-level scheme discussed is the central difference formula. This gives $u_{r,s+1}$ in terms of u at nodes on the time steps $s - 1$ and s. Other three-level formulas suitable for limited ranges of values of p are now proposed.

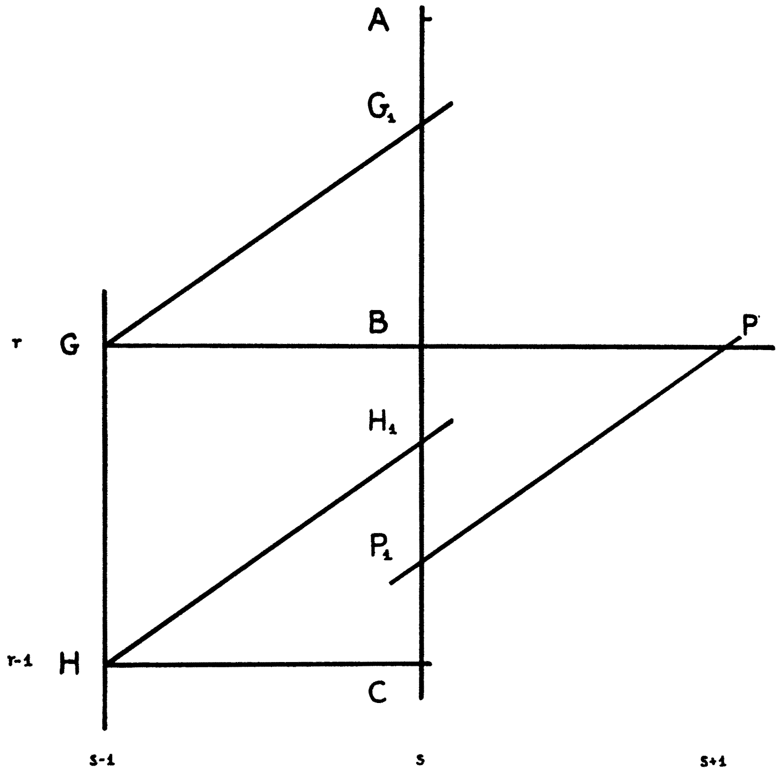

FIG. 6

Referring to Figure 6, PP_1, GG_1 and HH_1 are the characteristics through P, G, and H respectively, thus $u_P = u_{P_1}$, $u_G = u_{G_1}$ and $u_H = u_{H_1}$. If P_1 lies between B and C, and a cubic interpolation formula incorporating the values of u at G_1, B, H_1, and C is used to give the values of u at points between B and C, then the value of u at P is given by

$$
\begin{aligned}
u_{r,s+1} = &-\frac{(1-2p)(1-p)}{1+p}\, u_{r,s-1} + 2(1-2p)u_{r,s} \\
&+ 2pu_{r-1,s-1} - \frac{2p(1-2p)}{1+p}\, u_{r-1,s}.
\end{aligned}
\tag{10}
$$

In Figure 7, PP_1, HH_1, and II_1 are the characteristics through P, H, and I respectively, thus $u_P = u_{P_1}$, $u_H = u_{H_1}$, and $u_I = u_{I_1}$. If P_1 lies between B and C, and a cubic interpolation formula incorporating the values of u at B, H_1, C and I_1 is used to give the values of u at points between B and C, then the value of u at P is given by

$$
\begin{aligned}
u_{r,s+1} = &-\frac{2(2p-1)(1-p)}{2-p}\, u_{r,s} + 2(1-p)u_{r-1,s-1} \\
&+ 2(2p-1)u_{r-1,s} - \frac{p(2p-1)}{2-p}\, u_{r-2,s-1}.
\end{aligned}
\tag{11}
$$

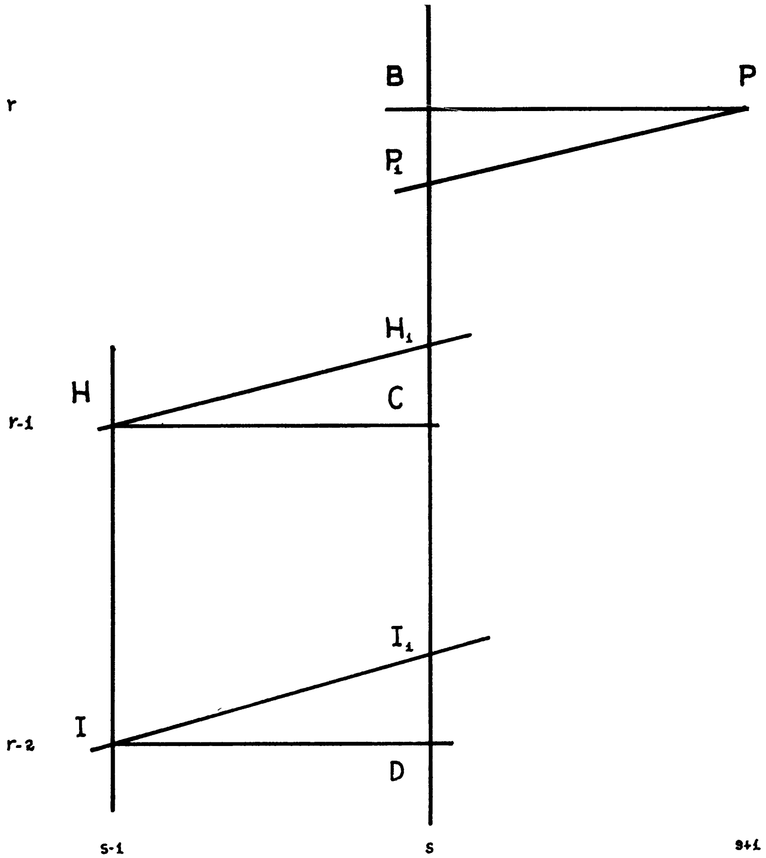

FIG. 7

The stability of (10) and (11) for the range $0 \leqq p \leqq 1$ can be demonstrated in the usual manner.

Numerical calculations are now carried out using selected two- and three-level interpolation schemes to solve (2). The results are shown in Tables 2(a) and 2 (b). The errors are shown in the last two rows where $\sum | e |$ is the sum of the moduli of the errors after a time $6 \frac{\Delta x}{V}$ and $\sum | e_1 |$ refers to the errors at a later stage in the computation when the profile has been transported over a further time $6 \frac{\Delta x}{V}$. In the case of the three-level formula (11), after a time $36 \frac{\Delta x}{V}$ the sums of the moduli of the errors are still only 0.660 and 0.026 for the "roof top" and "sine" curves respectively. The results shown in Tables 2(a) and 2(b) are for values of p lying between 0 and 1, but in the case of the two-level schemes they may be interpreted for values of p outside this range. For example, the figures for $p = \frac{1}{2}$ refer also to $p = n + \frac{1}{2}$ if the profile is moved on a further $12n$ intervals of x.

5. Interpolation Formulas and Finite Difference Schemes. In view of the form of the transport equation, a close link might be expected between interpolation

TABLE 2(a)

r	Correct Values	Scheme: Parabolic Formula (8)	Cubic Formula (9)	Three-Level Formula (10)	Three-Level Formula (11)
		p = $\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{3}{4}$
		s = 12	12	24	8
−4	0	0	0	0	0
−3	0	0.001	0	0	0
−2	0	−0.004	0	0	0
−1	0	−0.004	0	0	0
0	0	0.020	0	0	0
1	0	0.024	0.003	0	0
2	0	−0.045	−0.005	0	0
3	0	−0.126	−0.032	−0.003	0.001
4	0	−0.068	−0.026	−0.027	−0.011
5	0	0.165	0.121	0.063	0.038
6	0.5	0.502	0.471	0.471	0.479
7	1.0	0.961	0.961	1.003	1.000
8	1.5	1.591	1.505	1.498	1.499
9	2.0	2.248	2.068	2.007	1.997
10	2.5	2.651	2.554	2.555	2.522
11	3.0	2.682	2.757	2.872	2.923
12	2.5	2.433	2.554	2.557	2.541
13	2.0	2.006	2.068	1.992	1.999
14	1.5	1.452	1.505	1.501	1.500
15	1.0	0.876	0.961	0.996	1.001
16	0.5	0.421	0.471	0.472	0.488
17	0	0.156	0.121	0.063	0.038
18	0	0.043	−0.026	−0.028	−0.020
19	0	0.008	−0.032	0.003	0
20	0	0.001	−0.005	0	0
21	0	0	0.003	0	0
22	0	0	0	0	0
23	0	0	0	0	0
24	0	0	0	0	0
25	0	0	0	0	0
26	0	0	0	0	0
$\sum \|e\|$		1.838	1.007	0.509	0.287
$\sum \|e_1\|$		2.644	1.169	0.660	0.432

formulas and difference schemes used to solve (2). This is best illustrated by means of an example. Consider the problem of evolving a finite difference replacement of (2) which makes use of the points P, B, C, and D in Figure 5. Taylor expansions about the point B give

$$u_{r,s+1} = u_{r,s} + \Delta t \left(\frac{\partial u}{\partial t}\right)_{r,s} \tag{12}$$

$$u_{r-1,s} = u_{r,s} - \Delta x \left(\frac{\partial u}{\partial x}\right)_{r,s} + \tfrac{1}{2} (\Delta x)^2 \left(\frac{\partial^2 u}{\partial x^2}\right)_{r,s} \tag{13}$$

TABLE 2(b)

r	Correct Values	Scheme: Parabolic Formula (8)		Cubic Formula (9)	Three-Level Formula (10)	Three-Level Formula (11)
		p	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{3}{4}$
		s	12	12	24	8
0	−0.707		−0.733	−0.702	−0.706	−0.706
1	−0.924		−0.933	−0.917	−0.922	−0.923
2	−1.000		−0.992	−0.993	−0.999	−0.999
3	−0.924		−0.900	−0.917	−0.923	−0.923
4	−0.707		−0.670	−0.702	−0.706	−0.706
5	−0.383		−0.338	−0.380	−0.382	−0.382
6	0		0.044	0	0	0
7	0.383		0.420	0.380	0.382	0.382
8	0.707		0.733	0.702	0.706	0.706
9	0.924		0.933	0.917	0.922	0.923
10	1.000		0.992	0.993	0.999	0.999
11	0.924		0.900	0.917	0.923	0.923
12	0.707		0.670	0.702	0.706	0.706
13	0.383		0.338	0.380	0.382	0.382
14	0		−0.044	0	0	0
15	−0.383		−0.420	−0.380	−0.382	−0.382
$\sum \lvert e \rvert$			0.460	0.074	0.016	0.014
$\sum \lvert e_1 \rvert$			0.914	0.144	0.026	0.014

and

$$(14) \qquad u_{r-2,s} = u_{r,s} - 2\Delta x \left(\frac{\partial u}{\partial x}\right)_{r,s} + 2(\Delta x)^2 \left(\frac{\partial^2 u}{\partial x^2}\right)_{r,s}.$$

The value of $\left(\frac{\partial u}{\partial t}\right)_{r,s}$ is obtained from (12) and $\left(\frac{\partial u}{\partial x}\right)_{r,s}$ by eliminating $\left(\frac{\partial^2 u}{\partial x^2}\right)_{r,s}$ from (13) and (14). These values are then substituted into (2) to give

$$(15) \qquad u_{r,s+1} = \left(1 - \frac{3p}{2}\right) u_{r,s} + 2pu_{r-1,s} - \frac{p}{2} u_{r-2,s}.$$

The truncation error in (15) is dominated by the term $\frac{1}{2}(\Delta t)^2\left(\frac{\partial^2 u}{\partial t^2}\right)_{r,s}$ neglected in (12) and since by differentiating (2) the result

$$(16) \qquad \frac{\partial^2 u}{\partial t^2} = V^2 \frac{\partial^2 u}{\partial x^2}$$

is obtained, it follows that the principal part of the truncation error is $\frac{1}{2}p^2(\Delta x)^2 \frac{\partial^2 u}{\partial x^2}$. This is the standard finite difference approach which can, however, be improved in the following manner. Replace equation (12) by

$$(17) \qquad u_{r,s+1} = u_{r,s} + \Delta t \left(\frac{\partial u}{\partial t}\right)_{r,s} + \tfrac{1}{2} (\Delta t)^2 \left(\frac{\partial^2 u}{\partial t^2}\right)_{r,s},$$

which, on using (2) and (16) becomes

$$u_{r,s+1} = u_{r,s} - p\Delta x \left(\frac{\partial u}{\partial x}\right)_{r,s} + \tfrac{1}{2} p^2(\Delta x)^2 \left(\frac{\partial^2 u}{\partial x^2}\right)_{r,s}. \tag{18}$$

If $\left(\frac{\partial u}{\partial x}\right)_{r,s}$ and $\left(\frac{\partial^2 u}{\partial x^2}\right)_{r,s}$ are now eliminated from (13), (14), and (18), the parabolic interpolation formula (8) with $n = 1$ is obtained with truncation error

$$\tfrac{1}{6}p(1 - p)(2 - p)(\Delta x)^3 \frac{\partial^3 u}{\partial x^3}.$$

This is a distinct improvement over the previous finite difference formula (15), and in particular if p is close to unity, the interpolation formula is expected to be specially accurate when used to solve (2). If $p = 1$, of course, the theoretical solution of the interpolation formula (8) with $n = 1$ is the same as the theoretical solution of (2). However, as it is intended to use the results of the present investigation to solve the general transport equation (1), the exact correspondence of the theoretical solutions of (2) when $p = 1$ can really be ignored. This example illustrates the fact that the best finite difference formula for a given set of points used to solve (2) is an interpolation formula. This is because each derivative with respect to a

TABLE 3

Scheme	*Formula Number*	*Truncation Error*
Friedrichs	(4)	$\frac{1}{2}(1 - p^2)(\Delta x)^2 \frac{\partial^2 u}{\partial x^2}$
Carlson $0 \leqq p \leqq 1$	(5a)	$\frac{1}{2}p(1 - p)(\Delta x)^2 \frac{\partial^2 u}{\partial x^2}$
Carlson $p > 1$	(5b)	$\frac{1}{2}p(p + 1)(\Delta x)^2 \frac{\partial^2 u}{\partial x^2}$
Central Difference	(6)	$-\frac{1}{3}p(1 - p^2)(\Delta x)^3 \frac{\partial^3 u}{\partial x^3}$
Linear Interpolation	(7c)	$-\frac{1}{2}(n - p)(n - p + 1)(\Delta x)^2 \frac{\partial^2 u}{\partial x^2}$
Parabolic Interpolation	(8)	$-\frac{1}{6}(n - p - 1)(n - p)(n - p + 1)(\Delta x)^3 \frac{\partial^3 u}{\partial x^3}$
Cubic Interpolation	(9)	$\frac{1}{24}(n - p - 1)(n - p)(n - p + 1) \cdot (n - p + 2)(\Delta x)^4 \frac{\partial^4 u}{\partial x^4}$
Three-Level I	(10)	$-\frac{1}{2}p^2(1 - p)(1 - 2p)(\Delta x^4) \frac{\partial^4 u}{\partial x^4}$
Three-Level II	(11)	$\frac{1}{2}p(1 - p)^2(1 - 2p)(\Delta x)^4 \frac{\partial^4 u}{\partial x^4}$

co-ordinate is a constant multiple of the corresponding derivative with respect to the other co-ordinate, thus the Taylor expansions can all be expressed in terms of a single variable. Elimination of the maximum possible number of derivatives with respect to this variable leads to an interpolation formula.

6. Truncation Errors. For purposes of comparison, the truncation errors associated with the finite difference schemes considered for solving (2) are given in Table 3. The errors quoted are Δx times the errors as defined by Richtmyer [1, p. 19].

7. Conclusions. The calculations carried out in the present paper, using existing stable finite difference schemes in turn to solve the simplified transport equation (2), vary considerably in accuracy. The central difference formula (6) is most accurate with Carlson's scheme for $|p| \leqq 1$ next in order of merit. Carlson's implicit scheme for $|p| > 1$ is very poor, particularly for large values of $|p|$. This is illustrated in Figure 8 where the part of the truncation error depending on p is shown as a function (E) of p. It can be seen that the maximum value of the truncation error when $0 \leqq p \leqq 1$ is one-eighth of the minimum value when $p > 1$. In fact, the authors believe that implicit schemes can be abandoned as a means of obtaining accurate solutions of the transport equation.

New explicit schemes, derived as interpolation formulas, are next used to solve (2) and a considerable improvement in accuracy is obtained, particularly for schemes such as (9), (10), and (11), which are cubic interpolation formulas with a very small truncation error. The error in any numerical solution of (2) takes the form of a smoothing out of the initial profile together with, in most cases, a superposed stable oscillation.

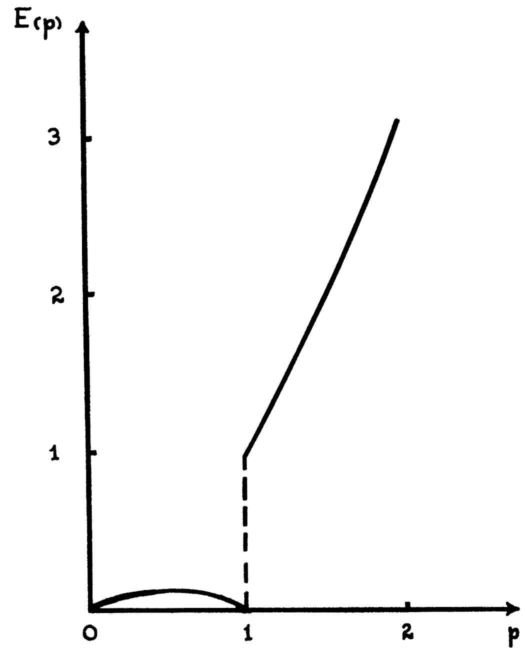

FIG. 8

It cannot be emphasized too strongly, however, that schemes which successfully solve (2) do not necessarily give comparable accuracy when used to solve (1), where V is a function of x, t, and u. On the other hand, difference schemes which fail to give accurate solutions of (2), can hardly be expected to be more successful when used to solve (1). The main difficulty in solving (1) numerically arises from the fact that the characteristics are curved and the distance BP_1 (Figures 1, 2, 3, 5, 6, 7) is no longer given simply by $V\Delta t$ or $p\Delta x$. It must be found by integrating the equation

$$\frac{dx}{dt} - V(x, t, u) = 0. \tag{19}$$

If, in the case of curved characteristics, BP_1 is now expressed as $p'\Delta x$, any one of the interpolation formulas proposed in the present paper may be applied directly with p' substituted for p. The value of p' is, of course, in general different at each node.

In deciding the values of Δx and Δt for a given calculation, Δx is first chosen to represent adequately the initial profile. The time step Δt is then chosen so that BP_1 is large enough for the calculation to proceed without too many interpolations but not so large that the positions of P_1, obtained from (19), are too much in error. We hope to examine in detail at a later date the general problem of integrating (1).

8. Acknowledgements. The calculations were performed on the Deuce computer of Glasgow University, and the authors are indebted to Dr. D. C. Gillies, Director of the Computing Laboratory, for permission to use the machine and to Miss A. Low for help in running the programs.

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